A Proof of the Elliptic-Function Addition-Theorem.

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The integration of the differential equation

$$\frac{d\phi}{\Delta\phi}+\frac{d\psi}{\Delta\psi}=0 \quad (\Delta\phi=\sqrt{1-k^2\sin^2\phi},\ \text{etc.})$$

can be effected by aid of the factor

$$\frac{\Delta\phi\Delta\psi-k^2\sin\phi\cos\phi\sin\psi\cos\psi}{1-k^2\sin^2\phi\sin^2\psi},$$

which, as will shortly be seen, is an integrating factor; thus,

$$\begin{aligned}
0&=\frac{\Delta\phi\Delta\psi-k^2\sin\phi\cos\phi\sin\psi\cos\psi}{1-k^2\sin^2\phi\sin^2\psi}\left(\frac{d\phi}{\Delta\phi}+\frac{d\psi}{\Delta\psi}\right)\\
&=\frac{1}{1-k^2\sin^2\phi\sin^2\psi}\left(\Delta\psi d\phi-k^2\sin\phi\cos\phi\sin\psi\cos\psi\frac{d\psi}{\Delta\psi}\right)+(\text{——});^*
\end{aligned}$$

and since $1-k^2\sin^2\phi\sin^2\psi=\cos^2\phi+\sin^2\phi-k^2\sin^2\phi\sin^2\psi=\cos^2\phi+\sin^2\phi\Delta^2\psi$,

$$\begin{aligned}
\text{this}\ &=\frac{1}{\cos^2\phi+\sin^2\phi\Delta^2\psi}\left(\Delta\psi d\phi-k^2\sin\phi\cos\phi\sin\psi\cos\psi\frac{d\psi}{\Delta\psi}\right)+(\text{——})\\
&=\frac{1}{1+\tan^2\phi\Delta^2\psi}\left(\Delta\psi\sec^2\phi d\phi-k^2\tan\phi\sin\psi\cos\psi\frac{d\psi}{\Delta\psi}\right)+(\text{——})\\
&=\frac{1}{1+\tan^2\phi\Delta^2\psi}\,d(\tan\phi\Delta\psi)+(\text{——})\\
&=d(\tan^{-1}\tan\phi\Delta\psi)+d(\tan^{-1}\tan\psi\Delta\phi)\\
&\therefore \tan^{-1}(\tan\phi\Delta\psi)+\tan^{-1}(\tan\psi\Delta\phi)=\mu;
\end{aligned}$$

i. e.

$$\frac{\tan\phi\Delta\psi+\tan\psi\Delta\phi}{1-\tan\phi\tan\psi\Delta\phi\Delta\psi}=\tan\mu.$$

And since evidently $\mu=\phi$ when $\psi=0$, μ is the amplitude of $(u+v)$, where u, v are the elliptic functions whose amplitudes are ϕ, ψ respectively.

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* (——) being the same function of (ψ,ϕ) that the preceding term is of (ϕ,ψ).

The formula for sn $(u+v)$ can be very readily derived from above; thus,

$$\operatorname{sn}(u+v) = \sin\mu = \frac{\tan\mu}{\sqrt{1+\tan^2\mu}}$$

$$= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{(\tan\phi\Delta\psi + \tan\psi\Delta\phi)^2 + (1 - \tan\phi\,\tan\psi\Delta\phi\Delta\psi)^2}}$$

$$= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{1 + \tan^2\phi\Delta^2\psi + \tan^2\psi\Delta^2\phi + \tan^2\phi\,\tan^2\psi\Delta^2\phi\Delta^2\psi}}$$

$$= \frac{\tan\phi\Delta\psi + \tan\psi\Delta\phi}{\sqrt{(1+\tan^2\phi\Delta^2\psi)(1+\tan^2\psi\Delta^2\phi)}}$$

$$= \frac{\sin\phi\,\cos\psi\Delta\psi + \cos\phi\,\sin\psi\Delta\phi}{\sqrt{(\cos^2\phi + \sin^2\phi\Delta^2\psi)(\cos^2\psi + \sin^2\psi\Delta^2\phi)}}$$

$$= \frac{\sin\phi\,\cos\psi\Delta\psi + \cos\phi\,\sin\psi\Delta\phi}{1 - k^2\sin^2\phi\,\sin^2\psi},$$

since $\quad \cos^2\phi + \sin^2\phi\Delta^2\psi = 1 - k^2\sin^2\phi\,\sin^2\psi = \cos^2\psi + \sin^2\psi\Delta^2\phi;$

i. e. $\quad \operatorname{sn}(u+v) = \operatorname{sn}u\,\operatorname{cn}v\,\operatorname{dn}v + \operatorname{sn}v\,\operatorname{cn}u\,\operatorname{dn}u \div 1 - k^2\operatorname{sn}^2u\,\operatorname{sn}^2v.$

The cn $(u+v)$ and dn $(u+v)$ can of course be just as readily obtained.